ON THE EXTENSION OF THE OPTICAL-MECHANICAL ANALOGY

(O PRODOLZHENII OPTIKO-MEKHANICHESKOI ANALOGII)

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N. G. CHETAEV
(Moscow)

Hamilton discovered the analogy between the wave-optics of Huygens and the motions of a mechanical system subject to holonomic constraints and subjected to forces that can be represented by a potential function.

This famous discovery has directed the progress of analytical dynamics for a whole century.

Theories of light continued to develop. Cauchy was the first to set the problem on the corresponding extension of the optical-mechanical analogy. In this article we establish the analogy between the mathematical theory of light of Cauchy and the stable motion of holonomic, conservative mechanical systems [1].

Let us investigate a mechanical system subject to holonomic constraints. Let us denote its independent generalized (holonomic) coordinates as $q_1, \ldots, q_n$, n is the number of degrees of freedom, $p_1, \ldots, p_n$ are the conjugate momenta.

For simplicity we will assume that the holonomic constraints are independent of time, and the forces acting on the system are represented by a potential function $U(q_1, \ldots, q_n)$ independent of time.

Let

$$2T = \sum_{i,j} g_{ij}\, p_i p_j$$

denote twice the kinetic energy of the material system under consideration; under assumptions made, functions $g_{ij} = g_{ji}$ do not depend on time t and may be dependent on the coordinates q_s.

Hamilton's partial differential equation has the form

$$\sum_{ij} g_{ij} \frac{\partial V}{\partial q_i} \frac{\partial V}{\partial q_j} = 2(U + h) \tag{1}$$

where h represents the constant of the kinetic energy.

The complete integral of the Hamilton equation (1) is a function

$$U\,(q_1, \ldots, q_n;\ \alpha_1 \ldots, \alpha_n) + \text{const},$$

satisfying equation (1) and depending on the constants $a_1, \ldots, a_n$ none of which is arbitrary:

$$\left\| \frac{\partial^2 V}{\partial q_i \partial \alpha_j} \right\| \neq 0$$

and the constant of the kinetic energy is some function of the constants a_s

$$h = h\,(\alpha_1, \ldots, \alpha_n)$$

According to the well-known theorem of Hamilton-Jacobi, the general solution of the equation of motion is given by equations

$$p_i = \frac{\partial V}{\partial q_i},\ \beta_i = -\,t\,\frac{\partial h}{\partial \alpha_i} + \frac{\partial V}{\partial \alpha_i} \qquad (i = 1, \ldots, n) \tag{2}$$

where β_i are constants.

Perturbed motions of the mechanical system are defined by different values of constants a_j and β_j.

In order to select from among the possible motions of a mechanical system those which are stable with respect to the variables, under conditions of perturbation of only the initial values, let us investigate the differential equations for Poincaré variations

$$\frac{d\xi_i}{dt} = \sum_j \left(\frac{\partial^2 H}{\partial q_j \partial p_i}\,\xi_j + \frac{\partial^2 H}{\partial p_j \partial p_i}\,\eta_j \right)$$

$$\frac{d\eta_i}{dt} = -\sum_j \left(\frac{\partial^2 H}{\partial q_j \partial q_i}\,\xi_j + \frac{\partial^2 H}{\partial p_j \partial q_i}\,\eta_i \right) \tag{3}$$

where ξ_i, η_i are variations of coordinates q_i and of momenta p_i, and

$$H = T - U$$

For a stable unperturbed motion, equations for Poincaré variations (3) represent a system of linear differential equations, reducible by means of a nonsingular linear transformation of variables to a system of linear differential equations with constant coefficients; all the characteristic values of the system of independent solutions must be equal to zero.

Variations of the coordinates and the momenta of those perturbed motions that are defined by variations of constants β_i only, whilst the a_i values remain fixed, will have zero characteristic values, if the unperturbed motion is stable.

In such perturbed motions, because of equation (2)

$$\eta_i = \sum_j \frac{\partial^2 V}{\partial q_i \partial q_j} \xi_j \qquad (i = 1, \ldots, n) \tag{D}$$

Hence, taking into consideration the explicit expression for H, we have

$$\frac{d\xi_i}{dt} = \sum_{j,s} \xi_s \frac{\partial}{\partial q_s}\left(g_{ij}\frac{\partial V}{\partial q_j}\right) \qquad (i = 1, \ldots, n) \tag{4}$$

Variables q_i and constants a, appearing in the right-hand part of equation (4), must be replaced by their values corresponding to an unperturbed motion.

For a stable unperturbed motion let equation (4) be reducible by a nonsingular linear transformation

$$x_i = \sum_j \gamma_{ij} \xi_j$$

with a constant determinant $\Gamma = \| \gamma_{ij} \|$.

If $\xi_{1r}, \ldots, \xi_{nr} (r = 1, \ldots, h)$ denote a normal system of independent solutions of equation (4), then

$$x_{ir} = \sum_j \gamma_{ij} \xi_{jr}$$

will be the solutions of the reduced system. For a stable unperturbed motion all the characteristic values of the solutions $(x_{1r}, \ldots, x_{nr})$ are zero, as we have seen, and consequently

$$\| x_{sr} \| = C^* = \| \gamma_{sj} \| \, \| \xi_{jr} \| = \Gamma C \exp \int \sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial V}{\partial q_j}\right) dt$$

where C^*, C are some constants different from zero. It follows from the last relation that for a stable perturbed motion

$$\sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial V}{\partial q_1}\right) = 0 \tag{5}$$

As the functions g_{ij} are defined by the expression for kinetic energy

$$2T = \sum g_{ij} p_i p_j$$

the principal diagonal minors of the discriminant $\| g_{ij} \|$ will all, according to the theorem of Sylvester, be positive, and consequently equation (5) will be elliptic.

Let us investigate some twice differentiable function

$$\Phi(-ht + V)$$

dependent on the complete integral $-ht + V$ of the Hamilton-Jacobi partial differential equation.

For a stable unperturbed motion, because of (5), (1) and (2) we have

$$\sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial \Phi}{\partial q_j}\right) = \Phi' \sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial V}{\partial q_j}\right) + \Phi'' \sum g_{ij}\frac{\partial V}{\partial q_i}\frac{qV}{\partial q_j} = \frac{2(U+h)}{h^2}\frac{\partial^2 \Phi}{\partial t^2}$$

and consequently

$$\frac{2(U+h)}{h^2}\frac{\partial^2 \Phi}{\partial t^2} = \sum \frac{\partial}{\partial q_i}\left(g_{ij}\frac{\partial \Phi}{\partial q_j}\right)$$

This wave equation establishes the analogy between the mathematical theory of light of Cauchy and the stable motions of holonomic conservative systems.

If, when integrating the Hamilton equation (1), the variables can be separated, then conditions of stability similar to (5) could be written down for each complete group of separated variables.

BIBLIOGRAPHY

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